AXIALLY-SYMMETRIC FORMS OF EQUILIBRIUM OF AN ELASTIC SPHERICAL SHELL UNDER UNIFORMLY DISTRIBUTED PRESSURE

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The assumption that the original state of a system is an ideal one lies at the basis of the stability analysis of elastic systems. For example, in the stability of elastic rods one assumes that the axis is straight, that the material is uniform and that the compressive forces are applied centrally. In the stability of plates, one assumes that initial deflection is absent and that the external forces are applied only at the middle surface.

In actuality, real systems differ from the ideal in one way or another. In most cases the difference does not lead to serious discrepancies between theory and experiment. The condition for the critical state of an ideal system differs only slightly from the condition for the excessive distortion of a real system.

Nevertheless, there are problems in which neglect of the differences from ideal have led to substantial errors. One such problem is that of the stability of a spherical shell loaded by uniformly distributed pressure (Fig. 1).

1. We assume that the real shell has a small deviation from the ideal. Let the deviation of the surface from the spherical form be specified for simplicity by a single parameter f.

Under the action of external pressure the shell acquires a local bending w. The nature of the relation $p = p(w)$ is determined by the value of f (Fig. 2).

The ordinate is the pressure p expressed as a nondimensional parameter

$$p^\circ = \frac{pR^2}{2Eh^2}$$

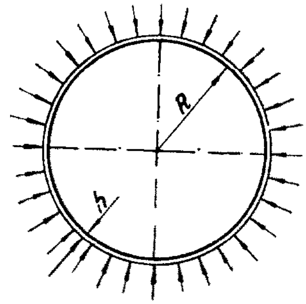

Fig. 1.

The loss of stability occurs suddenly. For example, for $f = f_3$ the pressure rises only up to the value $p^\circ(A)$ (at the point A), after which there occurs a sudden passage to a new form of equilibrium (point B). The pressure $p^\circ(A)$ increases with a reduction in the parameter f. With $f = 0$ we have

$$p^\circ = \frac{pR^2}{2Eh^2} = 0.606$$

that is, the value of the critical pressure given by the classical theory for an ideal system.

The small but real deviations of the system from the ideal show in this case as strong an effect on the equilibrium state as is shown by tests on the loss of stability, which indicate that

$$p^\circ(A) = 0.10 \text{ to } 0.15$$

instead of the expected $p_*^\circ = 0.606$. The shaded portion of Fig. 2 shows the probable zone of the loss of stability.

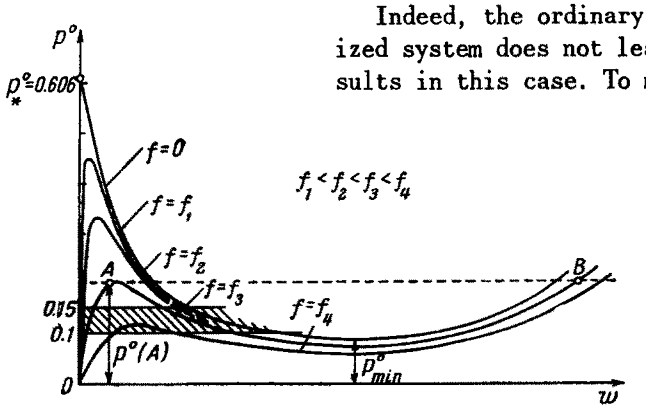

Fig. 2.

Indeed, the ordinary analysis of an idealized system does not lead to satisfactory results in this case. To make an exact determination of the critical pressure would require a statistical analysis of the errors arising from the preparation and testing of the shell. The solution of such a problem does not appear possible at the present time. There arises in this connection the concept of stability in the large, in accordance with which one may attack such problems from a statistical approach, limited as before to idealized systems but now in the region of large displacements. As a result of one such solution the curve $p^\circ = p^\circ(w)$ is determined in Fig. 2 for the case $f = 0$; the so-called lower critical pressure p°_{min} is found, which is a lower limit to the possible critical pressure. Loss of stability is considered to be probable within the interval

$$p^\circ_{\min} < p^\circ < p_*^\circ$$

Thus, there arises the problem of the determination of the form of equilibrium of a spherical shell in the region of large displacements.

The first attempt at a solution of this problem was made by von Kármán and Tsien [1]. The authors made the assumption that the bending of the shell was of a local nature and that the dimpled zone was included within small angles θ (Fig. 3). Other authors also found this assumption to be convenient. The problem was solved by an energy method in [1]. They considered that no shell deformation took place beyond the limits of the dimple and that within the dimple the circumferential strain ϵ_2 was zero. Conditions imposed later were unjustified and led to false results.

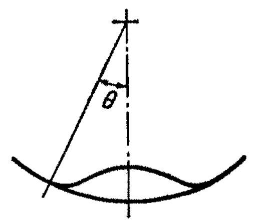

Fig. 3.

Many attempts to solve this problem were made following von Kármán and Tsien. Some authors (see, for example, [2] and [3]) used the variational methods of Ritz and Galerkin and invariably obtained values for the lower critical pressure of the order of $p^\circ_{\min}$ from 0.13 to 0.40, depending upon the shape of the function selected and upon the method used. It is clear that the pressure $p^\circ_{\min} = 0.13$ is too high, since experiments give $p^\circ(A)$ from 0.10 to 0.15, and $p^\circ_{\min}$ must be still smaller.

An attempt was made in 1954 to show that the value of $p^\circ_{\min}$ must be negative and that consequently there must exist forms of equilibrium of a dimpled spherical shell in the absence of pressure [4]. The problem was solved by the Galerkin method. The dimpled zone resisted the remaining part of the shell through an intermediate zone of local bending, with the geometric and force conditions fully maintained. The calculation gave $p^\circ_{\min} = -0.13$.

This work was criticized by Mushtari [5], who by a different variational method but with the same approximation function obtained $p^\circ_{\min} = +0.1$. These results were seen as proof of the high degree of sensitivity of the solution to the variational method used and to the choice of approximation function. It became clear that the use of the variational method is exhausted for this problem.

Thus the problem of the determination of the lower critical pressure has acquired not only a practical but also a theoretical interest.

The first attempts to apply variational methods to the solution of nonlinear equations of equilibrium for a spherical shell were made numerically by Keller and Reiss [6] and by Murray and Wright [7]. These papers solved the problem of the equilibrium of a shallow cupola clamped around the edge. Unjustifiably, the solution was carried over to the closed sphere. The force boundary conditions on the contour of the dimple do not agree with those in the unbent portion of the shell. The calculation provided for a choice of ratios of radius to thickness R/h.

The solution proposed is a numerical one for the nonlinear equations of a spherical cupola, obtained with the aid of high-speed machines. The solution is independent of the parameter R/h and is exact within the limits of applicability of the equations for shallow shells.

2. We take the dimple on the sphere to be of relatively small dimensions. It is therefore permissible to consider the shell as shallow in the dimpled zone. Such an approach is universally adopted today and leads to no contradictions.

The equations for a shallow spherical shell in the assumed axially-symmetric forms of equilibrium have the following form [4,8]:

$$\begin{aligned}\rho \frac{d^2\psi}{d\rho^2} + \frac{d\psi}{d\rho} - \frac{\psi}{\rho} &= \vartheta\left(\theta_1\rho + \frac{\vartheta}{2}\right)\\ \rho \frac{d^2\vartheta}{d\rho^2} + \frac{d\vartheta}{d\rho} - \frac{\vartheta}{\rho} &= -\lambda\psi(\theta_1\rho + \vartheta) + \nu\rho^2\end{aligned} \qquad (2.1)$$

Here

$$\rho = \frac{r}{r_1}, \qquad \psi = -\frac{T_1\rho}{Eh}, \qquad \lambda = 12(1-\mu^2)\frac{r_1^2}{h^2}, \qquad \nu = \lambda\frac{pr_1}{2Eh} \qquad (2.2)$$

where T_1 is the radial tensile force, ϑ the angle of rotation of the arc meridian, r_1 any arbitrarily fixed radius (Fig. 4), and θ_1 is the angle of slope of the shell at this radius.

In these equations one may set $\sin\theta \approx \tan\theta \approx \theta$, or $\theta^2 \ll 1$.

It is convenient to take for the value of r_1 the radius R of the sphere. Since $\theta_1 = r_1/R$, then $\theta_1 = 1$.

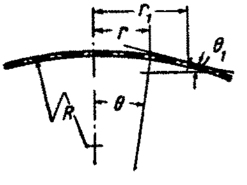

Fig. 4.

The region of applicability of Equations (2.1) is evidently determined by the ratio

$$\rho^2 = \frac{r^2}{R^2} \ll 1$$

In order to be free of the parameter R/h, we introduce new variables

x, Θ and Ψ in place of ρ, ϑ and ψ

$$\rho = \frac{x}{\lambda^{1/4}}, \quad \vartheta = \frac{\Theta}{\lambda^{1/4}}, \quad \psi = \frac{\nu}{\lambda}\rho + \frac{1}{\lambda^{3/4}}\Psi \tag{2.3}$$

The system of equations (2.1) then takes the form

$$x\frac{d^2\Psi}{dx^2} + \frac{d\Psi}{dx} - \frac{\Psi}{x} = \Theta\left(x + \frac{1}{2}\Theta\right)$$

$$x\frac{d^2\Theta}{dx^2} + \frac{d\Theta}{dx} - \frac{\Theta}{x} = -\Psi(x+\Theta) - p_0 x\Theta \tag{2.4}$$

where

$$p_0 = \frac{pR^2}{Eh^2}\sqrt{3(1-\mu^2)} \tag{2.5}$$

and in agreement with (2.2) and (2.3)

$$x = \frac{r}{R}\left[\frac{R^2}{h^2}12(1-\mu^2)\right]^{1/4}, \qquad \Theta = \vartheta\left[\frac{R^2}{h^2}12(1-\mu^2)\right]^{1/4}$$

$$\Psi = -\frac{r}{EhR}\left(T_1 + \frac{pR}{2}\right)\left[\frac{R^2}{h^2}12(1-\mu^2)\right]^{3/4} \tag{2.6}$$

Thus, the variable x represents a nondimensional radius, the function Θ gives a measure of the angle of rotation of a meridian arc on a certain scale, and the function Ψ shows that the radial tension T_1 differs from its value $pR/2$ in the momentless state. It follows from the first of Equations (2.6) that the system (2.4) holds for values

$$x \ll \left[\frac{R^2}{h^2}12(1-\mu^2)\right]^{1/4} \approx 1.8\sqrt{\frac{R}{h}} \tag{2.7}$$

We note that the proposed change in the equations is possible also for nonsymmetrical forms of equilibrium of spherical and cylindrical shells. Generally, it becomes necessary to represent the critical loads as a function of the parameter R/h, as is frequently done.

At the center of the dimple for $r = 0$ we have $\Theta = 0$ and $\Psi = 0$, and outside the limits of the dimple (at infinity) the momentless state must be conserved; i.e. $\Theta = 0$ and $\Psi = 0$ (since $T_1 = -pR/2$).

We divide the region of variation of x into two parts:

$$0 \leqslant x \leqslant x_k \text{ and } x_k \leqslant x < \infty.$$

Beyond x_k we take a high enough value of x so that the decaying function may be considered as negligibly small compared to x. It is clear that the choice of x_k depends upon the magnitude of the displacements assumed for the solution of the problem.

The system (2.4) is linearized for $x > x_k$ and takes the form

$$x\frac{d^2\Psi}{dx^2}+\frac{d\Psi}{dx}-\frac{\Psi}{x}=\Theta x \tag{2.8}$$

$$x\frac{d^2\Theta}{dx^2}+\frac{d\Theta}{dx}-\frac{\Theta}{x}=-\Psi x-p_0x\Theta \tag{2.9}$$

3. We construct first the solution in the region $x > x_k$.

Equation (2.8) is satisfied if one takes

$$\Theta = CH_1(\alpha x), \qquad \Psi = -\frac{C}{\alpha^2}H_1(\alpha x)$$

where $H_1(\alpha x)$ is the Hankel function of index one and α is an undetermined parameter. By substitution of Θ and Ψ in Equation (2.9) we get

$$\alpha^2 = \frac{p_0}{2} \pm \sqrt{\frac{p_0^2}{4}-1}$$

The smallest p_0 for which the parameter α has a real value will be

$$p_0 = 2$$

With this the external pressure p, for $\mu = 0.3$, takes on the upper critical value

$$p_*^0 = \frac{pR^2}{2Eh} = 0.606$$

which follows from Expression (2.5).

We are interested in the forms of equilibrium for p smaller than the upper critical value; i.e. for $p_0 < 2$. Therefore

$$\alpha^2 = \frac{p_0}{2} \pm i\sqrt{1-\frac{p_0^2}{4}}$$

From this we get

$$\alpha = \tfrac{1}{2}\sqrt{2+p_0} + i\tfrac{1}{2}\sqrt{2-p_0} \qquad \bar{\alpha} = \tfrac{1}{2}\sqrt{2+p_0} - i\tfrac{1}{2}\sqrt{2-p_0}$$

We have, correspondingly, two conjugate Hankel functions; then

$$\Theta = C_1H_1^{(1)}(\alpha x) + C_2H_1^{(1)}(\bar{\alpha}x)$$

$$\Psi = -C_1\frac{1}{\alpha^2}H_1^{(1)}(\alpha x) - C_2\frac{1}{\bar{\alpha}^2}H_1^{(1)}(\bar{\alpha}x) \tag{3.1}$$

Here the Hankel functions are taken to be of the first kind, vanishing at infinity. Let

$$H_1^{(1)}(\alpha x) = X_1 + iY_1, \qquad H_1^{(1)}(\bar{\alpha}x) = X_1 - iY_1 \tag{3.2}$$

$$C_1 = \tfrac{1}{2}(C-Di), \qquad C_2 = \tfrac{1}{2}(C+Di) \tag{3.3}$$

Then

$$\Theta = CX_1 + DY_1 \tag{3.4}$$

$$\Psi = -C\left(\frac{p_0}{2}X_1 + \sqrt{1-\frac{p_0^2}{4}}\,Y_1\right) - D\left(\frac{p_0}{2}Y_1 - \sqrt{1-\frac{p_0^2}{4}}\,X_1\right) \tag{3.5}$$

The derivatives of Θ and Ψ will be required. In accordance with the rules for differentiation of cylindrical functions we obtain from Expressions (3.1)

$$\frac{d\Theta}{dx} = C_1\alpha H_0^{(1)}(\alpha x) + C_2\bar{\alpha} H_0^{(1)}(\bar{\alpha}x) - \frac{1}{x}\Theta$$

$$\frac{d\Psi}{dx} = -\frac{C_1}{\alpha}H_0^{(1)}(\alpha x) - \frac{C_2}{\bar{\alpha}}H_0^{(1)}(\bar{\alpha}x) - \frac{1}{x}\Psi$$

Here $H_0^{(1)}(\alpha x)$ is the Hankel function of the first kind and zero order. By making use of the notation in (3.2) and (3.3) we obtain

$$\frac{d\Theta}{dx} = C\left[\frac{\sqrt{2+p_0}}{2}X_0 - \frac{\sqrt{2-p_0}}{2}Y_0 - \frac{1}{x}X_1\right] + D\left[\frac{\sqrt{2+p_0}}{2}Y_0 + \right.$$
$$\left. + \frac{\sqrt{2-p_0}}{2}X_0 - \frac{1}{x}Y_1\right] \tag{3.6}$$

$$\frac{d\Psi}{dx} = C\left[-\frac{\sqrt{2+p_0}}{2}X_0 - \frac{\sqrt{2-p_0}}{2}Y_0 + \frac{1}{x}\frac{p_0}{2}X_1 + \frac{1}{x}\sqrt{1-\frac{p_0^2}{4}}\,Y_1\right] +$$
$$+ D\left[-\frac{\sqrt{2+p_0}}{2}Y_0 + \frac{\sqrt{2-p_0}}{2}X_0 + \frac{1}{x}\frac{p_0}{2}Y_1 - \frac{1}{x}\sqrt{1-\frac{p_0^2}{4}}\,X_1\right] \tag{3.7}$$

in which by analogy with Expressions (3.2) we denote

$$H_0^{(1)}(\alpha x) = X_0 + iY_0, \qquad H_0^{(1)}(\bar{\alpha}x) = X_0 - iY_0 \tag{3.8}$$

The displacements along the normal to the middle surface of the shell are, taking account of (2.3)

$$w = \int_r^\infty \vartheta\, dr, \quad \text{or} \quad w = \frac{R}{\lambda^{1/2}}\int_x^\infty \Theta\, dx \tag{3.9}$$

Upon substitution of Θ from (3.1) we get

$$w = -\frac{R}{\lambda^{1/2}}\left[\frac{C_1}{\alpha}H_0^{(1)}(\alpha x) + \frac{C_2}{\bar{\alpha}}H_0^{(1)}(\bar{\alpha}x)\right] \tag{3.10}$$

and upon passing to real functions

$$\frac{w}{h}\sqrt{12(1-\mu^2)} = C\left[\frac{\sqrt{2+p_0}}{2}X_0 + \frac{\sqrt{2-p_0}}{2}Y_0\right] + D\left[\frac{\sqrt{2+p_0}}{2}Y_0 - \frac{\sqrt{2-p_0}}{2}X_0\right]$$

Thus, the solution of the problem in the linear region is obtained in X_1, Y_1, X_0, Y_0 functions, which themselves represent the real and imaginary parts of a Hankel function with a complex argument. These functions are not tabulated and computational methods are required.

We shall assume that the value of x_k is large enough not only for non-linearity to be neglected but also large enough to permit the use of asymptotic expansions.

The Hankel function of the first kind and of order c has the form in an asymptotic expansion

$$H_c^{(1)}(z) = \sqrt{\frac{2}{\pi z}} \exp\left[i\left(z - \frac{1}{2}\pi c - \frac{1}{4}\pi,\right]\sum_{m=0,1,2,\ldots} \frac{(-1)^m (c, m)}{(2iz)^m}\right.$$

$$\left((c, m) = \frac{(4c^2-1^2)(4c^2-3^2)\ldots[4c^2-(2m-1)^2]}{m!\,2^m}\right)$$

For calculation of the functions X_1 and Y_1, the function must be separated into real and imaginary parts, setting $c = 1$ and $z = ax$. For the functions X_0 and Y_0 one must take $c = 0$.

We get after transformation

$$X_1 = \frac{e^{-bx}}{\sqrt{\pi x}}\left\{\frac{b}{\sqrt{1-a}}\left[P_1\cos\left(ax - \frac{3\pi}{4}\right) - Q_1\sin\left(ax - \frac{3\pi}{4}\right)\right] + \right.$$
$$\left. + \sqrt{1-a}\left[P_1\sin\left(ax - \frac{3\pi}{4}\right) + Q_1\cos\left(ax - \frac{3\pi}{4}\right)\right\}\right.$$

$$Y_1 = \frac{e^{-bx}}{\sqrt{\pi x}}\left\{\frac{b}{\sqrt{1-a}}\left[Q_1\cos\left(ax - \frac{3\pi}{4}\right) + P_1\sin\left(ax - \frac{3\pi}{4}\right)\right] + \right.$$
$$\left. + \sqrt{1-a}\left[Q_1\sin\left(ax - \frac{3\pi}{4}\right) - P_1\cos\left(ax - \frac{3\pi}{4}\right)\right]\right\}$$

$$X_0 = \frac{e^{-bx}}{\sqrt{\pi x}}\left\{\frac{b}{\sqrt{1-a}}\left[P_0\cos\left(ax - \frac{\pi}{4}\right) - Q_0\sin\left(ax - \frac{\pi}{4}\right)\right] + \right.$$
$$\left. + \sqrt{1-a}\left[P_0\sin\left(ax - \frac{\pi}{4}\right) + Q_0\cos\left(ax - \frac{\pi}{4}\right)\right]\right\}$$

$$Y_0 = \frac{e^{-bx}}{\sqrt{\pi x}}\left\{\frac{b}{\sqrt{1-a}}\left[Q_0\cos\left(ax - \frac{\pi}{4}\right) + P_0\sin\left(ax - \frac{\pi}{4}\right)\right] + \right.$$
$$\left. + \sqrt{1-a}\left[Q_0\sin\left(ax - \frac{\pi}{4}\right) - P_0\cos\left(ax - \frac{\pi}{4}\right)\right]\right\} \qquad (3.11)$$

Here

$$a = \frac{1}{2}\sqrt{2+p_0}, \qquad b = \frac{1}{2}\sqrt{2-p_0}$$

$$P_1 = 1 + \frac{3\sqrt{2-p_0}}{8x} + \frac{15p_0}{64x^2} - \frac{105(1+p_0)\sqrt{2-p_0}}{256x^3} + \frac{4725(1-{}^1/_2\,p_0^2)}{2048x^4} +$$
$$+ \frac{72765(-1+p_0+p_0^2)\sqrt{2-p_0}}{16384x^5} - \frac{2837835}{1024}\,\frac{p_0(3-p_0^2)}{128x^6} + \ldots$$

$$Q_1 = \frac{3\sqrt{2+p_0}}{8x} - \frac{15\sqrt{4-p_0^2}}{64x^2} + \frac{105(1-p_0)\sqrt{2+p_0}}{256x^3} + \frac{4725p_0\sqrt{4-p_0^2}}{4096x^4} -$$

$$-\frac{72765\,(1+p_0-p_0^2)\sqrt{2+p_0}}{16384x^5}+\frac{2837835}{1024}\,\frac{(1-p_0^2)\sqrt{4-p_0^2}}{128x^6}+\dots$$

$$P_0=1-\frac{\sqrt{2-p_0}}{8x}-\frac{9p_0}{64x^2}+\frac{75\,(1+p_0)\sqrt{2-p_0}}{256x^3}-\frac{3675\,(1-{}^1/_2\,p_0^2)}{2048x^4}-$$
$$-\frac{59535\,(-1+p_0+p_0^2)\sqrt{2-p_0}}{16384x^5}+\frac{2401245}{1024}\,\frac{p_0\,(3-p_0^2)}{128x^6}+\dots$$

$$Q_0=-\frac{\sqrt{2+p_0}}{8x}+\frac{9\sqrt{4-p_0^2}}{64x^2}-\frac{75\,(1-p_0)\sqrt{2+p_0}}{256x^3}-\frac{3675p_0\sqrt{4-p_0^2}}{4096x^4}+$$
$$+\frac{59535\,(1+p_0-p_0^2)\sqrt{2+p_0}}{16384x^5}-\frac{2401245}{1024}\,\frac{(1-p_0^2)\sqrt{4-p^{20}}}{128x^6}+\dots \qquad (3.12)$$

4. We return now to the "nonlinear" part $0 \leqslant x \leqslant x_k$. The region of variation of x is bounded on the right. In addition, only one parameter p_0 remains, and so there is the possibility of numerical integration of the system joining up with the solution at the boundary $x = x_k$. By introduction of the notation

$$\frac{d\Theta}{dx}=u, \qquad \frac{d\Psi}{dx}=v \qquad (4.1)$$

we rewrite the system (2.4) in finite differences

$$\Delta v=\left[\Theta+\frac{\Theta^2}{2x}-\frac{v}{x}+\frac{\Psi}{x^2}\right]\Delta x$$
$$\Delta u=\left[-\Psi-\frac{\Psi\Theta}{x}-p_0\Theta-\frac{u}{x}+\frac{\Theta}{x^2}\right]\Delta x \qquad (4.2)$$
$$\Delta\Theta=u\Delta x, \qquad \Delta\Psi=v\Delta x$$

The functions Ψ and Θ must be zero at $x = 0$. The functions u and v remain indeterminate at $x = 0$. By postulating a series of values of u_0 and v_0 and integrating from $x = 0$ to $x = x_k$, we choose u_0 and v_0 so that the continuity of the functions on the boundary is assured.

At $x = x_k$ there are evidently four conditions to be satisfied

$$\Theta_k=\Theta_k, \qquad \left.\frac{d\Theta}{dx}\right|_k=\left.\frac{d\Theta}{dx}\right|_k \qquad (4.3)$$

$$\Psi_k=\Psi_k, \qquad \left.\frac{d\Psi}{dx}\right|_k=\left.\frac{d\Psi}{dx}\right|_k \qquad (4.4)$$

where on the one hand the functions are found from numerical integration of the nonlinear system and on the other hand are determined from Expressions (3.4) to (3.7) in the linear system.

Condition (4.3) determines the equality of angles and moments on the boundary, and condition (4.4) determines the equality of meridian forces and displacements in the middle surface.

Upon returning to Expressions (3.4) to (3.10) and (4.1), we obtain the continuity condition in the following form:

$$\Theta_k = CX_{1k} + DY_{1k}$$

$$\Psi_k = -C\left[\frac{p_0}{2}X_{1k} + \sqrt{1-\frac{p_0^2}{4}}\,Y_{1k}\right] - D\left[\frac{p_0}{2}Y_{1k} - \sqrt{1-\frac{p_0^2}{4}}\,X_{1k}\right]$$

$$u_k = C\left[\frac{\sqrt{2+p_0}}{2}X_{0k} - \frac{\sqrt{2-p_0}}{2}Y_{0k} - \frac{1}{x_k}X_{1k}\right] +$$

$$+ D\left[\frac{\sqrt{2+p_0}}{2}Y_{0k} + \frac{\sqrt{2-p_0}}{2}X_{0k} - \frac{1}{x_k}Y_{1k}\right]$$

$$v_k = C\left[-\frac{\sqrt{2+p_0}}{2}X_{0k} - \frac{\sqrt{2-p_0}}{2}Y_{0k} + \frac{1}{x_k}\frac{p_0}{2}X_{1k} + \frac{1}{x_k}\sqrt{1-\frac{p_0^2}{4}}\,Y_{1k}\right] +$$

$$+ D\left[-\frac{\sqrt{2+p_0}}{2}Y_{0k} + \frac{\sqrt{2-p_0}}{2}X_{0k} + \frac{1}{x_k}\frac{p_0}{2}Y_{1k} - \frac{1}{x_k}\sqrt{1-\frac{p_0^2}{4}}\,X_{1k}\right]$$

The constants C and D are determined from the first two equations and are substituted in the other two. Finally, we obtain the following two equations:

$$K_1\Theta_k + K_2\Psi_k - u_k = 0, \qquad -K_2\Theta_k + K_3\Psi_k - v_k = 0 \tag{4.5}$$

where

$$K_1 = \frac{(X_{0k}Y_{1k} - X_{1k}Y_{0k})(1-p_0)\sqrt{2+p_0} + (Y_{1k}Y_{0k} + X_{1k}X_{0k})(1+p_0)\sqrt{2-p_0}}{(X_{1k}^2 + Y_{1k}^2)\sqrt{4-p_0^2}} - \frac{1}{x_k}$$

$$K_2 = \frac{-(X_{0k}Y_{1k} - X_{1k}Y_{0k})\sqrt{2+p_0} + (Y_{1k}Y_{0k} + X_{1k}X_{0k})\sqrt{2-p_0}}{(X_{1k}^2 + Y_{1k}^2)\sqrt{4-p_0}}$$

$$K_3 = \frac{(X_{0k}Y_{1k} - X_{1k}Y_{0k})\sqrt{2+p_0} + (X_{0k}X_{1k} + Y_{0k}Y_{1k})\sqrt{2-p_0}}{(X_{1k}^2 + Y_{1k}^2)\sqrt{4-p_0^2}} - \frac{1}{x_k}$$

By making use of the asymptotic expansions (3.11) and (3.12), one may express the last equations in the form

$$K_1 = \frac{1}{\sqrt{2-p_0}}\left[-(1-p_0) - \frac{3}{2x_k^2} + \frac{63}{8x_k^4} + \frac{27\sqrt{2-p_0}}{x_k^5} + \frac{1899(1-p_0)}{16x_k^6} + \ldots\right]$$

$$K_2 = \frac{1}{\sqrt{2-p_0}}\left[1 - \frac{3}{2x_k^2} + \frac{3\sqrt{2-p_0}}{x_k^3} - \frac{63(1-p_0)}{8x_k^4} - \frac{27p_0\sqrt{2-p_0}}{x_k^5} + \right.$$

$$\left. + \frac{1899(1+p_0-p_0^2)}{16x_k^6}\ldots\right] \tag{4.6}$$

$$K_3 = \frac{1}{\sqrt{1-p_0}}\left[-1-\frac{3(1-p_0)}{2x_k^2}-\frac{3p_0\sqrt{2-p_0}}{x_k^3}+\frac{63(1+p_0-p_0^2)}{8x_k^4}-\right.$$
$$\left.-\frac{27(1-p_0^2)\sqrt{2-p_0}}{x_k^5}+\frac{1899(1-2p_0-p_0^2+p_0^3)}{16x_k^6}+\ldots\right]$$

5. The following numerical examples are presented.

We specify the value of p_0 in advance as that pressure for which it is necessary to find possible forms of equilibrium for a sphere with an elastic dimple. Since we are interested in pressures less than the upper critical pressure, $p_0 \leqslant 2$.

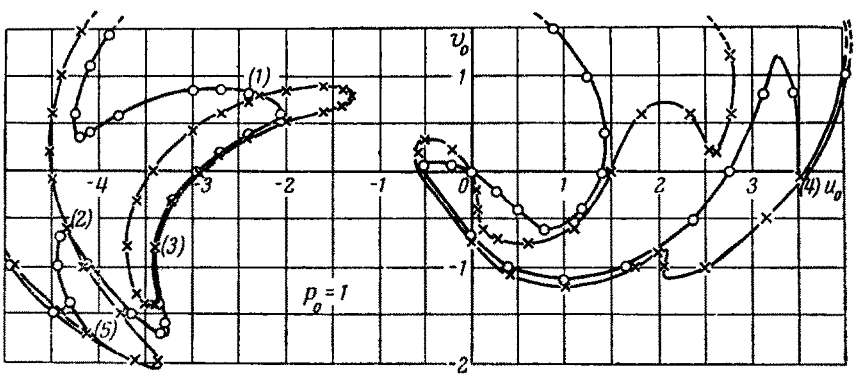

Fig. 5.

In addition we also fix the value of x_k. On one hand, this value must be large enough for all the nonlinearities of the problem to be contained within the interval $0 \leqslant x \leqslant x_k$; it must also be large enough for application of the conditions of asymptotic expansion. On the other hand it is desirable to keep it small in order not to increase the numerical work unduly.

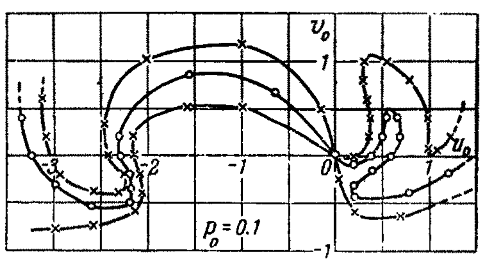

Fig. 6.

The value $x_k = 10$ was taken in the calculations.

In certain cases the choice of a large value of x_k may lead to contradictions with the basic assumptions applicable to the equations of shallow shells. It is easy to take into account the limits of variation in x for corresponding values of R/h.

From the assumed values of p_0 and x_k we calculate the coefficients K_1, K_2 and K_3 from Formulas (4.6). For example, for $p_0 = 1$ and $x_k = 10$ we have $K_1 = -0.01445$, $K_2 = 0.9878$, $K_3 = -1.0022$. Then, by taking a series of values $d\Theta/dx = u_0$, $d\Psi/dx = v_0$ for $x = 0$, we proceed with the numerical integration from $x = 0$ to $x = x_k$ according to (4.2). As a result of the integration we find values of u_k, v_k, Θ_k and Ψ_k, which may then be substituted in Equations (4.5). To find the decaying solution we select those values of u_0 and v_0 which satisfy the two equations of (4.5) simultaneously.

An electronic calculating machine was used in this work.

Tentative trials were made at first. They showed that for a fixed value of p_0 there is a multiplicity of integrals of Equations (4.2) for different values of u_0 (from -7 to $+7$) and v_0 (from -3 to $+2$) with intervals $\Delta u_0 = \Delta v_0 = 0.2$. The integration proceeded with steps of $\Delta x = 0.1$.

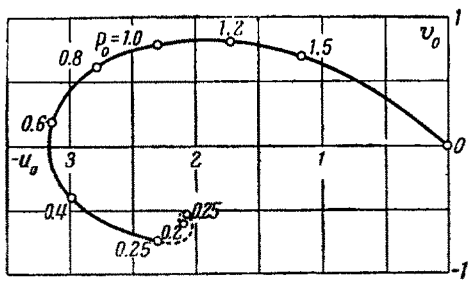

Fig. 7.

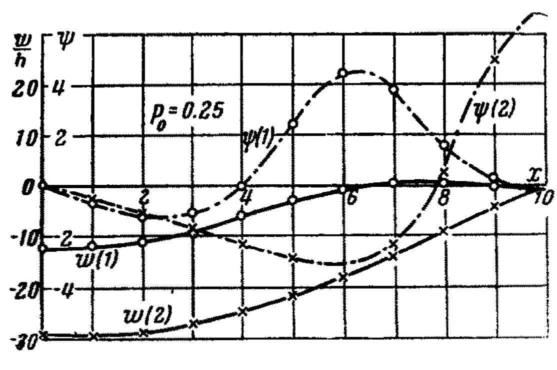

Fig. 8.

As a result of the preliminary trials u_0 and v_0 were determined as the values which reduced the left-hand sides of Equations (4.5) to zero. Two families of curves are shown by the crosses and the circular points of Fig. 5. Their intersection determines the unknown roots u_0 and v_0. These curves are shown in Fig. 5 for $p_0 = 1$ ($p^\circ = 0.303$) and the bracketed numbers denote roots. Corresponding curves for $p_0 = 0.1$ ($p^\circ = 0.0303$) are shown in Fig. 6. The points of intersection of the curves (exclusive of the zero point) were not observed.

Next, the interval of variation of u_0 and v_0 was contracted as determined by the distribution of roots, and the coordinates of the points of intersection were obtained more exactly. The calculation of the roots must be carried out to the 4th or 5th significant figure. Figure 7 shows the distribution of the basic groups of roots.

The final integration of Equation (4.2) provides more exact values of u_0 and v_0 with the functions obtained. Integration of the function Θ

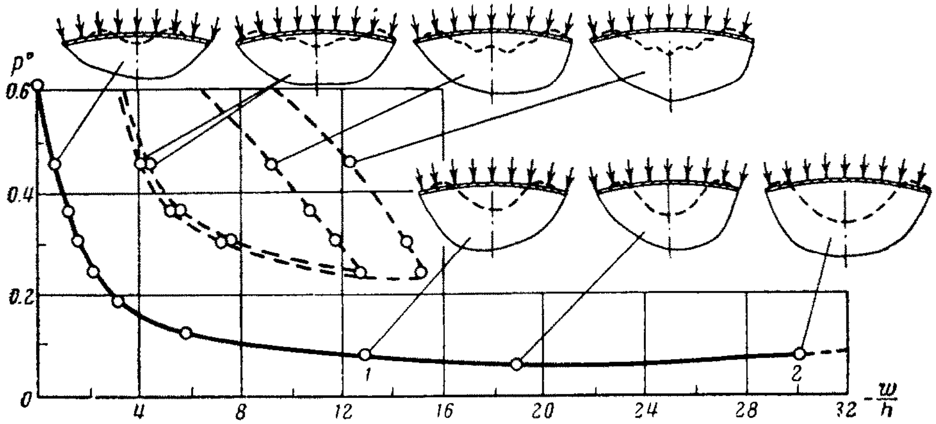

Fig. 9.

obtained in accordance with (3.9) gives the form of the elastic surface and the value of the maximum deflection. Figure 8 shows an example of the functions Ψ and w/h for two roots with $p_0 = 0.25$ ($p^\circ = 0.071$).

The solution obtained was subjected to a number of control checks.

In particular, the numerical integration was carried out beyond the limit x_k. The functions obtained coincide with the functions found from the linear system.

The calculation was made for values of $x_k > 10$. The solution does not depend on the value of x_k. This is confirmed by the fact that for $x_k = 10$ all nonlinearity is within the interval $0 < x < x_k$.

Finally, for improvement in the accuracy of the solution, the integration step was reduced to 0.01 with $\Delta x = 0.1$, which led to changes in the unknown function only in the 3rd figure. Further refinement was not necessary.

Figure 9 shows a summary of the results obtained. The continuous curve corresponds to the basic form of equilibrium. Numbers 1 and 2 show the points for which the functions Ψ and w/h were given in Fig. 8. The lower critical pressure is $p^\circ_{min} = 0.06$ at a value of relative deflection $w/h = 22$ to 23.

The broken lines in Fig. 9 show the curves corresponding to the higher forms of equilibrium.

The value of p°_{min} obtained will be as exact in the same measure as the applicability of the equations for shallow shells. This does not exclude a change in the value when a more exact system of equations is used.

BIBLIOGRAPHY

1. Von Kármán, T. and Tsien, H.S., The buckling of spherical shells by external pressure. *Jour. Aero. Sci.* Vol. 7, No. 2, 1939.

2. Friedrichs, K.O., On the minimum buckling load for spherical shells. *Von Kármán Anniversary Volume*, p. 258. Cal. Inst. Tech., 1941.

3. Mushtari, Kh.M. and Surkin, R.G., O nelineinoi teorii ustoichivosti uprugogo ravnovesiia tonkoi sfericheskoi obolochki pod deistviem ravnomerno raspredelennogo normal'nogo vneshnego davleniia (On the nonlinear theory of elastic stability of a thin spherical shell loaded by uniformly distributed external pressure). *PMM* Vol. 14, No. 6, 1950.

4. Feodos'ev, V.I., Ob ustoichivosti sfericheskoi obolochki, nakhodiashcheisia pod deistviem vneshnego ravnomerno raspredelennogo davleniia (On the stability of a spherical shell loaded by uniformly distributed external pressure). *PMM* Vol. 18, No. 1, 1954.

5. Mushtari, Kh.M., K teorii ustoichivosti sfericheskoi obolochki pod deistviem vneshnego davleniia (On the theory of the stability of a spherical shell loaded by external pressure). *PMM* Vol. 19, No. 2, 1955.

6. Keller, H.B. and Reiss, E.L., Spherical cap snapping. *Jour. Aero/Space Sci.* Vol. 26, No. 10, 1959.

7. Murray, F.J. and Wright, F.W., The buckling of thin spherical shells. *Jour. Aero/Space Sci.* Vol. 28, No. 3, 1961.

8. Feodos'ev, V.I., O bol'shikh progibakh i ustoichivosti krugloi membrany s melkoi gofrirovkoi (On the large deflections and stability of a circular membrane with fine corrugations). *PMM* Vol. 9, No.5, 1945.

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